PERIODIC AND APERIODIC BEHAVIOR IN DISCRETE ONEDIMENSIONAL DYNAMICAL SYSTEMS

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JEAN-MICHEL GRANDMONT

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INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
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by

Jean-Michel Grandmont*

1. Introduction

The theory of onedimensional nonlinear difference equations underwent considerable progress in recent years, as the result of the efforts of theorists from several fields - in particular from physics - to get a better understanding, by making use of the notion of the "Hopf's bifurcation," of the appearance of cycles and of the transition to aperiodic or "chaotic" behaviour in physical, biological or ecological systems. These new developments seem to be potentially very useful for the study of periodic and aperiodic phenomena in economics. Parts of this theory have been indeed already used in economic or game theory by Benhabib and Day [1981, 1982], Dana and Malgrange [1981], Day [1982, 1983], Grandmont [1983], Jensen and Urban [1982], Rand [1978].

The aim of this paper is to present some of these new developments in a compact form which will be, it is hoped, useable by economic theorists. The emphasis will be on the mathematical results of the theory, rather than on its possible applications.[1/]

Our basic reference will be Collet and Eckmann's book [1980] -

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thereafter denoted "CE." In order to simplify the presentation, we shall use in a few places stronger assumptions than in CE's book, which means that the reader interested in the more general (but more complicated) case and who wishes to look for complements will have to go back to their book. The definitions and the statements of the results will be self-contained. However, in the proofs of a few facts, we shall use freely the concepts introduced by CE, but we shall indicate where to find the appropriate definitions in that book.[2/]

2. Onedimensional Nonlinear Difference Equations

We are concerned thereafter with the difference equation $x_{t+1} = f(x_t)$, in which f is a function that maps the interval $[a,b]$ into itself. The object of the theory is the study of the existence (and the stability) of periodic solutions of this difference equation. To this effect, one defines recursively the iterates of f by $f^o(x) = x$ for all x (f^o is the identity map), $f^1 = f$ and $f^i = f\, f^{i-1}$. The orbit of x is then the set $\{x, f(x), f^2(x), \ldots\}$, which is composed of all iterates of x. The orbit is periodic if the cardinality of this set, say k, is finite, and its period is given by k. Equivalently, a periodic orbit or a cycle of f with (primitive) period k is defined by $(x_1, \ldots, x_k)$ such that 1) $f^k(x_1) = x_1$ and 2) $f^{i-1}(x_1) = x_i \neq x_1$ for $i = 2, \ldots, k$. This implies that all points x_i of the cycle are fixed points of f^k and that they all differ (one says then that x_i is a periodic point of f with period k).

Of course, if f is arbitrary, there is little hope to get interesting results. The simplifying feature of the theory is to assume that f is unimodal. More precisely, we say that f is unimodal if

1) f is continuous

2) there exists x^* in (a,b) such that f is increasing on $[a,x^*]$ - i.e., $f(x) > f(x')$ for all x, x' in $[a,x^*]$ such that $x > x'$ - and decreasing on $[x^*,b]$

3) $f(x^*) = b$

We shall say that f is C^1-unimodal if in addition

4) f is once continuously differentiable and $f'(x) \neq 0$ when $x \neq x^*$.

Note that when f is unimodal, then f has a unique fixed point $\bar{x}$ in the interval (x^*,b). Moreover, since f is decreasing on $[x^*,b]$ one has $f(b) < \bar{x} < b$ (see Figure 1.a). Finally, remark that the assumption that f is defined on a closed interval is not as restrictive as it may appear at first sight, since one may often go back to that case. For instance, if f maps the interval $[a,+\infty)$ into itself and is unimodal with a unique maximum at $x^* > a$, with $f(x^*) > x^*$, one may restrict attention without any loss of generality to the behaviour of f on the interval $[a,f(x^*)]$ since $f(x)$ belongs to that interval for any $x \geqq a$ (see Figure 1.b).[3/]

3. Sarkovskii's Theorem

We remarked earlier that when f is unimodal, it has a unique fixed point $\bar{x}$ in the interval (x^*,b). This fixed point is thus bound

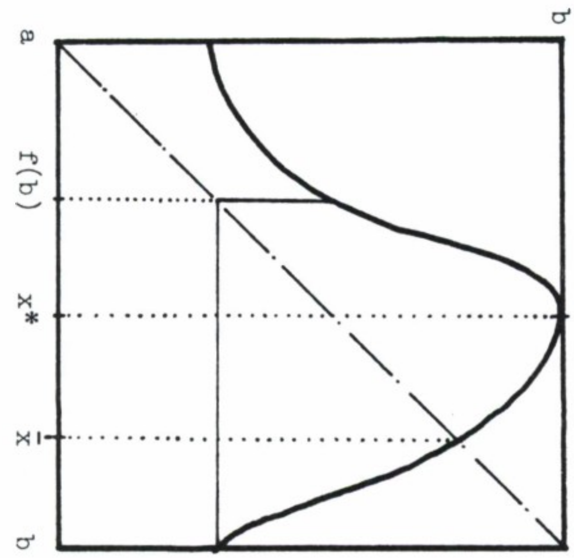

Figure 1.a

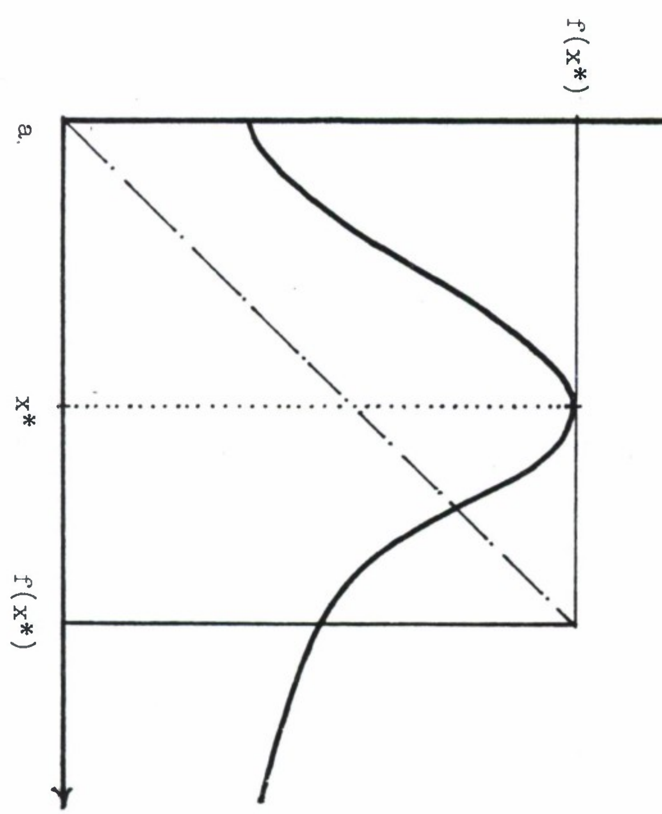

Figure 1.b

to coexist with any other periodic orbit. It turns out that one may get much more information concerning the coexistence of cycles displaying different periods. This is achieved in the following beautiful result, which is due to Sarkovskii [1964] - see also Stefan [1977].

Theorem 1: (Sarkovskii). Consider the ordering of the integers

$$
\begin{aligned}
& 3 > 5 > 7 \ldots \\
& > 2\cdot 3 > 2\cdot 5 > 2\cdot 7 > \ldots \\
& \quad \ldots \\
& > 2^n\cdot 3 > 2^n\cdot 5 > 2^n\cdot 7 > \ldots \\
& \quad \ldots \\
& > \ldots > 2^m > \ldots > 8 > 4 > 2 > 1 \quad .
\end{aligned}
$$

That is, first the odd integers greater than or equal to 3 forward, then the powers of 2 times these odd integers, and then the powers of 2 backward. If f is unimodal and has a cycle with period k then it has a cycle of period k' for every $k' < k$ in the sense of the above ordering.

Proof: This is (CE, Theorem II.3.10, p. 91). Q.E.D.

4. Stable Cycle

The preceding theorem implies that a unimodal map may have a lot of different cycles - think of the case in which f has a cycle of period 3. Some (or all) of them may be unstable, however, and thus essentially irrelevant as far as the dynamic behaviour of the system is concerned. It is therefore important to know how many stable cycles - if any - the map f possesses. It is only recently that a real

breakthrough was achieved on this matter by Singer [1978], who discovered that a unimodal map with a negative "Schwarzian derivative" could have at most one stable cycle.[4/]

Let us first define stability. Given the map f from $[a,b]$ into itself, consider a periodic orbit $(x_1,\dots,x_k)$. Since x_1 is a fixed point of f^k, we may say that this periodic orbit is (locally) stable if there exists an open neighborhood U of x_1 such that for every x in U, $f^{kt}(x)$ stays in U for all $t \geqq 1$ and $\lim_{t\to\infty} f^{kt}(x) = x_1$. When f is continuous, this implies that $f^{kt}(f^{i-1}(x))$ converges to x_1 as well for every $i = 2,\dots,k$. If f is continuously differentiable, this means that the derivative of f^k at x_1 has a modulus less than 1, i.e., $|Df^k(x_1)| < 1$. Of course, in order to make any sense, this definition should not depend upon the point chosen on the periodic orbit. As a matter of fact, we have by the chain rule of differentiation

$$\begin{aligned} Df^k(x_1) &= f'(x_k)Df^{k-1}(x_1) = \dots = f'(x_k)\ \dots\ f'(x_1) \\ &= Df^k(x_i) \end{aligned}$$

When f is continuously differentiable, we may therefore say that the cycle $(x_1,\dots,x_k)$ is stable if $|Df^k(x_1)| < 1$. The cycle will be said to be weakly stable if $|Df^k(x_1)| \leqq 1$ (this definition allows for "onesided" stability only).[5/] Finally, it will be said to be superstable if $Df^k(x_1) = 0$. When f is C^1-unimodal, this means that the critical point x^* belong to the periodic orbit.

We define next the notion of a Schwarzian derivative. Assume that f is thrice continously differentiable. The Schwarzian derivative of f at x, denoted $Sf(x)$, is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2$$

whenever $f'(x) \neq 0$. Direct computation shows that $Sf = -2|f'|^{1/2}D^2[|f'|^{-1/2}]$. So the condition that "f has a negative Schwarzian derivative" ($Sf < 0$ at every x such that $f'(x) \neq 0$) means that $|f'|^{-1/2}$ is convex on every interval of monotony of f. It will be satisfied in particular if $|f'|$ (or $\mathrm{Log}\,|f'|$) is concave on such intervals. But these sufficient conditions are by no means necessary. Finally the reader will note that the concavity of f is neither necessary nor sufficient to guarantee $Sf < 0$. Consider next the following conditions

S1. f is C^1-unimodal

S2. f is thrice continuously differentiable

S2. $Sf(x) < 0$ for all x in $[a,b]$, $x \neq x^*$.

Then we have

<u>Theorem 2</u>: Assume that f satisfies S1, S2, S3, $f(x) > x$ for all x in (a,x^*), and $f'(a) > 1$ whenever $f(a) = a$. Then

1) The map f has at most one weakly stable periodic orbit. This periodic orbit lies in the interval $[f(b),b]$.

2) If f has a weakly stable periodic orbit, it attracts the critical point x^*, that is, it coincides with the set of accumulation points of the sequence $(f^t(x^*))$.

Proof: We may note incidentally that under S1, S2, S3, one has $f(x) > x$ for all x in (a,x^*) whenever $f'(a) > 1$. This follows from the fact that since $Sf < 0$, f' cannot have a positive local minimum on that interval (see Step 3 of the proof of Theorem II.4.1 in CE, p. 97. Indeed, if there existed x in (a,x^*) such that $f(x) \leqq x$, then by the mean value theorem there would be y_1, y_2, with $a < y_1 \leqq x \leqq y_2 < x^*$ such that $f'(y_1) \leqq 1 < f'(y_2)$ and f' would be a positive local minimum in (a,y_2), a contradiction.

Remark now that when f is unimodal, $f(x) > x$ for all x in (a,x^*) implies that

(i) f maps the interval $[f(b),b]$ into itself (onto if and only if $f(b) \leqq x^*$)

(ii) for every x in $(a,f(b))$, there exists j such that $f^j(x) \in [f(b),b]$.

This follows from elementary considerations that are left to the reader. This shows that all periodic orbits - with the possible exception of an unstable fixed point of f at $x = a$ - must lie in $[f(b),b]$. In particular, any weakly stable cycle belongs to that interval.

Corollary II.4.2 in CE implies therefore that the statements of Theorem 2 are valid provided that f satisfies the additional condition

S4. f maps $[f(b),b]$ onto itself.

However, a closer look at CE's proof of this Corollary shows that it is still valid if S4 is replaced by the weaker

S4'. f maps the interval $[f(b),b]$ into itself.

But we have seen that this condition was implied by the assumptions of Theorem 2. The proof is complete. Q.E.D.

We shall note for further reference

S4". (i) $f(x) > x$ for all x in (a,x^*)

(ii) $f'(a) > 1$ when $f(a) = a$.

As we have seen, if f is unimodal, then S4" implies S4', while it implies S4 if and only if $f(b) \leqq x^*$.

The foregoing result provides an "experimental" way of verifying if a particular map satisfying the conditions of the theorem possesses a weakly stable cycle. It suffices indeed to check if the iterates of the critical point $f^t(x^*)$ converge to some periodic orbit and then to verify that the limit cycle is weakly stable. All these operations can in fact be easily achieved by using modern computers.

Maps that do not posses any weakly stable cycle appear to be good candidates to portray "chaotic" (aperiodic) behaviour in onedimensional dynamical systems. Theorem 2 provides a way to recognize whether or not a particular map is chaotic in the sense. Indeed, if f satisfies S1, S2, S3 and S4", then all cycles of f will be unstable if the iterates

of the critical point $f^t(x^*)$ do not converge or if they converge to an unstable periodic orbit. Again these conditions are easy to verify with the help of modern computers. Of course, since iterations must be stopped after a finite time in practice, this experimental way of proceeding will be unable to distinguish between chaotic behaviour and the presence of a weakly stable cycle that has a long period or that is only weakly attracting.

The next statement provides a condition involving the trajectory of the critical point x^* of f only, that ensures the existence of a (unique) weakly stable cycle. To this effect, we introduce some notation. Given a unimodal map f, for every x in $[a,b]$, the extended itinerary of x describes how the iterates $f^t(x)$ behave qualitatively, i.e., whether or not they fall on the right or on the left of the critical point x^*. More precisely, this extended itinerary $I_E(x)$ is an infinite sequence of R's, of L's and of C's obeying the following rule. If $[I_E(x)]_j$ denotes the j-th element of $I_E(x)$ for $j = 0,1,\ldots$, then $[I_E(x)]_j = R$ if $f^j(x) > x^*$, $[I_E(x)]_j = C$ if $f^j(x) = x^*$, and $[I_E(x)]_j = L$ if $f^j(x) < x^*$. We shall say that $I_E(x)$ is periodic with (primitive) period k if $[I_E(x)]_{j+k} = [I_E(x)]_j$ for all j and if k is the smallest integer having this property.

<u>Proposition 3</u>: Assume that f satisfies S1, S2, S3, S4" and

S5. $f''(x^*) < 0$.

Then f has a (unique) weakly stable cycle P if and only if the extended itinerary of the endpoint b, i.e., $I_E(b)$, is periodic. If the

period of $I_E(b)$ is k, the period of P is k or $2k$.

Proof: Assume that $I_E(b)$ has period k. If $f(b) \leqq x^*$, then S4 is satisfied, and from the "if" part of (CE, Proposition II.6.2), f has a weakly stable cycle in $[f(b),b]$. If $f(b) > x^*$, then $f(b) \leqq f^j(x^*)$ for all $j \geqq 1$. But it is then easy to verify that the restriction of f to $(f(b),b)$ has a sink in the sense of (CE, p. 107). Therefore from (CE, Lemma II.5.1), f has a weakly stable periodic orbit in $[f(b),b]$ in that case too (one can alternatively prove directly that f^2 has a weakly stable fixed point $[f(b),b]$, see the proof of Proposition 4). In all cases the weakly stable cycle is unique from Theorem 2. Finally, the fact that its period is k or $2k$ is an immediate consequence of (CE, Lemma II.3.2).

Assume conversely that f has a (unique) weakly stable cycle P of period k. It must lie in $[f(b),b]$. We wish to apply the "only if" part of (CE, Proposition II.6.2). A close look at their argument shows that their result is valid if S4 is replaced by S4' - and thus under S4" - but that it is correct only when the rightmost point of P, say x, satisfies $x \geqq x^*$ - which is the case under S1, S2, S3, S4', if and only if $k \geqq 2$ or when the periodic orbit is a fixed point in (x^*,b). The "only if" part of (CE, Proposition II.6.2) is not correct however under their assumptions if P is a weakly stable fixed point x of f such that $x < x^*$ (counterexamples are provided by making symmetric the cases 1-4 of Figure II.8 in CE, p. 102).[6/] The latter circumstance is ruled out however under S4", so the "only if" part of (CE, Proposition II.6.2) is valid under our assumptions. Thus $I_E(b)$ is periodic, and

from (CE, Lemma II.3.2), its period is k or $k/2$. Q.E.D.

The concept of (weak) stability that we have used is only local. It is thus important to know how large is the basin of attraction of a given weakly stable cycle. The next result states that under the conditions of Proposition 3, if there exists a weakly stable periodic orbit, which is then unique, the set of points that are not attracted to it is "exceptional."

Proposition 4: Assume that f satisfies S1, S2, S3, S4" and S5, and that it has a weakly stable cycle P. Let E be the set of points x in $[a,b]$ such that $f^t(x)$ does not tend to P. Then E has Lebesgue measure 0.

Proof: If $f(b) \leqq x^*$, S4 is satisfied. Then from (CE, Proposition II.5.7), the set E_f of points in $[f(b),b]$ that are not attracted to the weakly stable periodic orbit P, has Lebesgue measure 0.[7/] Let E_f' be the set of points x in $[a,f(b))$ such that $f^t(x) \in E_f$ for some t. Since f is increasing on $[a,x^*)$, the Lebesgue measure of E_f' is also 0. The set of points of $[a,b]$ that are not attracted to P is $E_f \cup E_f'$, to which one must add the endpoint a whenever $f(a) = a$, which shows the result in that case.

The case in which $x^* < f(b)$ is even simpler. The unique weakly stable cycle P belongs to $[f(b),b]$. Moreover, the iterates of any point x of $A = [x^*,b]$ lie in A, and oscillate around the unique fixed point $\bar{x}$ of f that belongs to A (whenever $x \neq \bar{x}$). In

particular, $I_E(b) = R^\infty$ and thus from Proposition 3, the period of P is 1 or 2. It is clear that $Df^2(x^*) = 0$ and $Df^2(x) > 0$ for all x in A, $x \neq x^*$. Furthermore, f^2 has a negative Schwarzian derivative on $(x^*,b]$ and has finitely many fixed point in $[x^*,b]$ - see steps 2 and 4 of the proof of Theorem II.4.1 in CE, pp. 97-98. Consider first the case in which the period of P is 1. Then since $f^2(x^*) = f(b) > x^*$, $f^2(b) < b$ and $Df^2(\bar{x}) \leqq 1$, one must have $f^2(x) > x$ for all x in $[x^*,\bar{x})$ and $f^2(x) < x$ for all x in $(\bar{x},b]$, otherwise there would be another weakly stable periodic orbit (of period 2). Thus $f^{2j}(x)$, and thus $f^j(x)$, converges to $\bar{x}$ as j tends to $+\infty$ for all x in $[x^*,b]$, see Figure 2.a. The other case in which the period of P is 2 its dealt with similarly. Let x_1 and x_2 be the two points of P. They satisfy $x^* < x_1 < \bar{x} < x_2 < b$. From the uniqueness of the weakly stable cycle, we have $Df^2(\bar{x}) > 1$ and in fact $f^2(x) > x$ for all x in (x^*,x_1) or $(\bar{x},x_2)$, and $f^2(x) < x$ for every x in $(x_1,\bar{x})$ or (x_2,b), see Figure 2.b. Thus $f^{2j}(x)$, and thus $f^j(x)$, converges to P as j tends to $+\infty$ for all x in $[x^*,b]$ except $x = \bar{x}$.

Thus if the period of P is 1, it attracts the whole interval $[a,b]$, except a if $f(a) = a$. If the period of P is 2, it attracts again the whole interval $[a,b]$, with the exception of the preimages of $\bar{x}$, i.e., of all points x of $[a,x^*)$ such that $f^j(x) = \bar{x}$ for some j, and of the endpoint a when $f(a) = a$. In the two cases, the exceptional set is finite or countable, which completes the proof. Q.E.D.

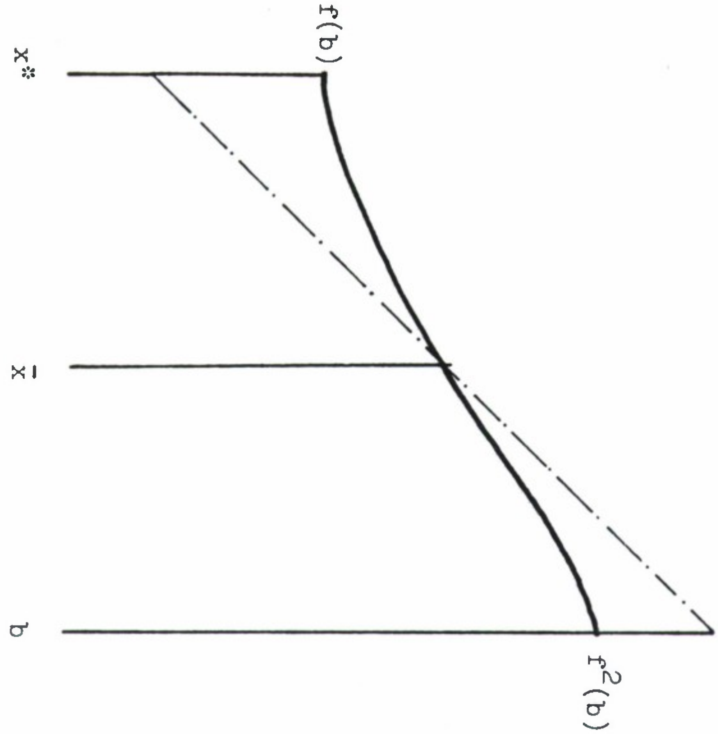

Figure 2.a

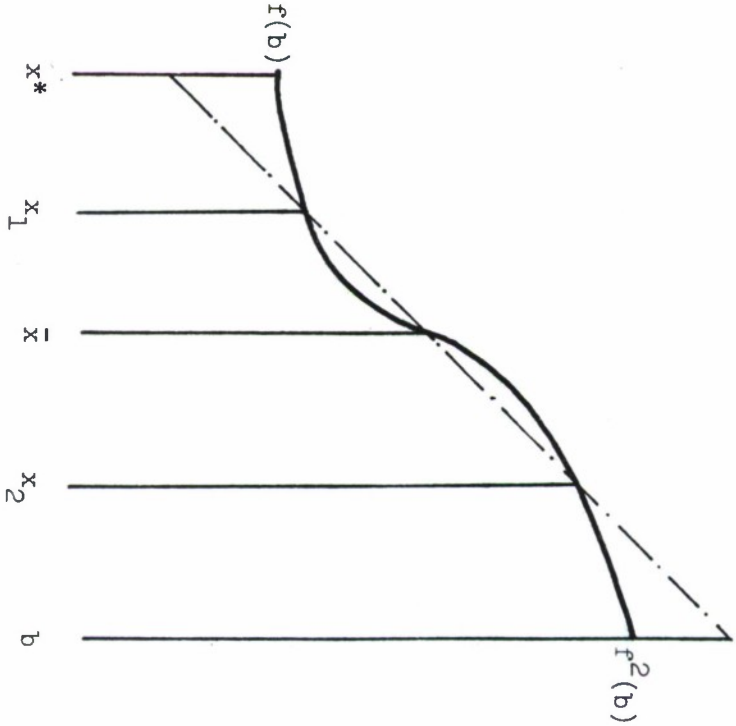

Figure 2.b

Remark: Proposition 4 shows that some claims according to which "period 3 implies chaos" are not always warranted. For instance, a consequence of the results of Li and Yorke [1975] is that if f is unimodal and if there exists a cycle of period 3, then there is an uncountable set S in $[a,b]$ and an $\varepsilon > 0$ such that for every x and y in S

$$\limsup_{j \to \infty} |f^j(x) - f^j(y)| \geqq \varepsilon$$

and

$$\liminf_{j \to \infty} |f^j(x) - f^j(y)| = 0$$

Thus trajectories with initial points in S - which may be called the "chaotic" set - come arbitrarily close and then noticeably separated infinitely often.

Some theorists have used this result (or a variant of it) to claim that the existence of a cycle of period 3 was an indication of chaotic behaviour (see in particular in economics Benhabib and Day [1981, 1982], Day [1982, 1983]). Proposition 4 shows that such a claim is unwarranted, for if there is a stable cycle, then the "chaotic" set S may be of Lebesgue measure 0 (think of a Cantor set) and thus essentially unobservable.

A more appropriate definition of chaos or aperiodicity is as we have seen, the property that all cycles are unstable.

5. Aperiodic Dynamics

As we said, maps f that have no weakly stable cycles appear to be good candidates to describe turbulent on "chaotic" behavior in one-dimensional dynamical systems. There is an obvious reason to look at such maps from that viewpoint. For if one considers a map f on $[a,b]$ that satisfies Assumptions S1, S2, S3, and if it has no weakly stable cycle, then for "most" initial points x the iterates of x, $f^j(x)$, will not display any periodic behaviour even if we wait long enough. Indeed under these assumptions, we know that f^k has only finitely many fixed points in $[a,b]$ (see steps 2 and 4 of the proof of Theorem II.4.1 in CE, pp. 97-98). Thus f has at most a countable number of cycles.8/ This implies that if E is the set of all points in $[a,b]$ that belong to a periodic orbit of f, E has Lebesgue measure 0, and that the orbit of any point x not in E is aperiodic, even if one iterates it long enough.

Among the class of such aperiodic maps, of special interest are those which possess a unique invariant probability measure which is absolutely continuous with respect to the Lebesgue measure, and which is ergodic. The probability measure ν on $[a,b]$ (endowed with its Borel σ-algebra) is said to be invariant with respect to f if $\nu(f^{-1}(A)) = \nu(A)$ for any Borel set. It is absolutely continuous with respect to the Lebesgue measure λ (absolutely continuous for short) if for any Borel set A, $\lambda(A) = 0$ implies $\nu(A) = 0$ (ν has then a λ-integrable density with respect to λ). Finally, ν is said to be ergodic if for any ν-integrable real-valued function g,

$$\frac{1}{n} \sum_{j=1}^{n} g(f^{j-1}(x)) \to \int g d\nu$$

as n tends to $+\infty$, for ν-almost every x. This implies in particular that if one considers for each x and every n, the empirical distribution $\nu_n(x)$ that is generated by the iterates $f^j(x)$ for $j = 0, \ldots, n-1$, which assigns probability $1/n$ to each $f^j(x)$, then the sequence $\nu_n(x)$ converges weakly to ν for ν-almost every x.[9/] Thus if ν is absolutely continuous and ergodic, although a given trajectory may look somewhat erratic since the iterates fill up eventually the support of the limit distribution ν, empirical distributions and time averages become ultimately fairly stable for ν-almost every initial point.

The next result gives a sufficient condition for the existence of a unique absolutely continuous invariant measure, which is ergodic.

Theorem 5: assume that f satisfies S1, S2, S3, S5, that is has no weakly stable periodic orbit, and that there exists an open neighbourhood V of x^* such that $f^j(x^*) \notin V$ for $j \geq 1$. Then f has a unique absolutely continuous invariant probability measure. It is ergodic.

Proof: Note first that if all cycles of f are unstable, S1, S2, S3 imply S4", otherwise f would have a weakly stable fixed point in $[a,x^*]$. Second, one must have $f(b) \leq x^*$, so that S4 is satisfied, otherwise f would have a weakly stable cycle in $[x^*,b]$. Thus we may apply (CE, Theorem III.8.3). Q.E.D.

Corollary 6: If f satisfies S1, S2, S3, S4", S5 and if the iterates
$f^j(x^*)$ of the critical point converge to an unstable cycle, then f has a unique absolutely continuous invariant probability measure. It is ergodic.

Proof: In view of Theorem 2, f has no weakly stable cycle and the iterates of x^* stay at a finite distance of x^*. Thus Theorem 5 applies. Q.E.D.

Remark: the foregoing results go in the direction of showing that aperiodic maps (having only unstable cycles) may display strong statistical regularities after all. Another direction of research has been to show that some (but not all) aperiodic maps may generate trajectories that are very sensitive to a small variation of initial conditions, thereby exhibiting the kind of phenomena that are observed e.g. in turbulent flows (maps that have a unique weakly stable periodic orbit as in Theorem 2 do not have such a sensitivity to initial conditions). For an aperiodic and sensitive map, a small error of measurement of the initial state, for instance, may result in very large prediction errors (relatively speaking) for future dates, even if the forecaster knows very well the law of motion of the system (the map f). For various definitions of sensitivity and a discussion of their implications, see (CE, pp. 15-22, 30-35, and Section II.7).

6. Topological Conjugacy

There is nothing intrinsic in the representation of a one-dimensional dynamical system by a particular difference equation $x_{t+1} = f(x_t)$, since one can always make a change of coordinates. We investigate now what happens when one makes a change of variable $y = h(x)$, in which h maps $[a,b]$ onto $[a',b']$, is once continuously differentiable, and $h'(x) > 0$ for all x in $[a,b]$. with the new variable, the dynamical system is represented by a new function g which maps $[a',b']$ into itself and satisfies $g(y) = h[f(h^{-1}(y))]$. Thus $g = h \circ f \circ h^{-1}$, we say then that f and g are topological conjugates.[10]

The maps f and g describe the same dynamics since the iterates of f and g are linked by $g^j = h\ f^j\ j^{-1}$ for all $j \geqq 0$. In particular $(x_1,\ldots,x_k)$ is a cycle of f if and only if $(h(x_1),\ldots,h(x_k))$ is a cycle of g. By differentiation one gets for all x

$$Dg^k(h(x))h'(x) = h'(f^k(x))Df^k(x)$$

and thus $Dg^k(h(x_i)) = Df^k(x_i)$ at any point of the periodic orbit. Stability or unstability of a periodic orbit is topologically invariant.

It is now immediate that S1 is topologically invariant, in the sense that f satisfies this condition if and only if g does. The same is true of S2 if h is thrice continuously differentiable. Conditions like $f(x) > x$ are also topologically inveriant, as well as S4, S4' or S4". Finally, it is easily seen that the condition

$f''(x^*) < 0$ is unchanged through the change of variable provided one assumes h to be twice continuously differentiable. Differentiating $g(h(x)) \equiv h(f(x))$ twice and evaluating the expressions at $x = x^*$ yields indeed

$$g''(h(x^*))[h'(x^*)]^2 = f''(x^*)h'(f(x^*))$$

if one takes into account the fact that $g'(h(x^*)) = f'(x^*) = 0$.

However, the condition S3 - which says in effect that $D^2|f'(x)|^{-1/2}$ is positive for all x in $[a,b]$, $x \neq x^*$ - is not generally invariant (like any convexity statement) through a (nonlinear) change of variable. The point of this discussion is that even when a particular map f does not satisfies S3, the foregoing results, i.e., Theorem 2 through Corollary 6, are still valid provided that one of the topological conjugates g of the original map f satisfies the assumptions made in anyone of these statements.

7. Bifurcations: Period Doubling and the Transition to Turbulance

Numerical experimentation with onedimensional nonlinear dynamical systems yields remarkable regularities that do not appear to depend much upon the maps under consideration. More precisely, consider a family of onedimensional unimodal maps f_λ that depend upon some real number λ, that may be thought as indexing one of the characteristics of the system (the parameter may be for instance under the control of some outside observer in a physical experiment). If we look back at Theorem 1, we should expect that the fashion in which cycles appear when λ is

varying, should display some degree of conformity with Sarkovskii's ordering of the integers. Namely, we should expect cycles having a period that is a power of 2 to appear first. Numerical experimentation shows that this is indeed the case. In fact, this is true for (weakly) stable cycles.

Let us assume that for each λ, we iterate the critical point x_λ^* of f_λ on a computer. If each f_λ satisfies the conditions of Theorem 2 and has in particular a negative Schwarzian derivative, we know that this procedure permits discovering (weakly) stable cycles that have a small period and that are attracting enough. Suppose now that we put λ on an horizontal axis and that above each value of λ we plot vertically the values taken by the iterates $f_\lambda^t(x_\lambda^*)$ for, say, $t = 200$ to 300. Computer simulations of this type yield typically a very neat "bifurcation diagram" which displays first a whole interval in which period doubling bifurcations occur more and more rapidly, a stable fixed point giving rise to a stable cycle of period 2, which yields then a stable cycle of period 4 and so on. The values of λ for which such period doubling bifurcations occur tend to some limit value λ_∞^*, beyond which one enters the "chaotic" region for $\lambda > \lambda_\infty^*$, one often observes a "mess" - meaning that one has either an aperiodic ("chaotic") map or a stable cycle with a very long period - in the middle of which windows may appear that show stable cycles with low periods like 3, 5, 6 or 7 (that depends of course of the degree of resolution of the diagram).[11]

The results that follow explain why such an outcome should be typically observed. Formally, we consider a one-parameter family of

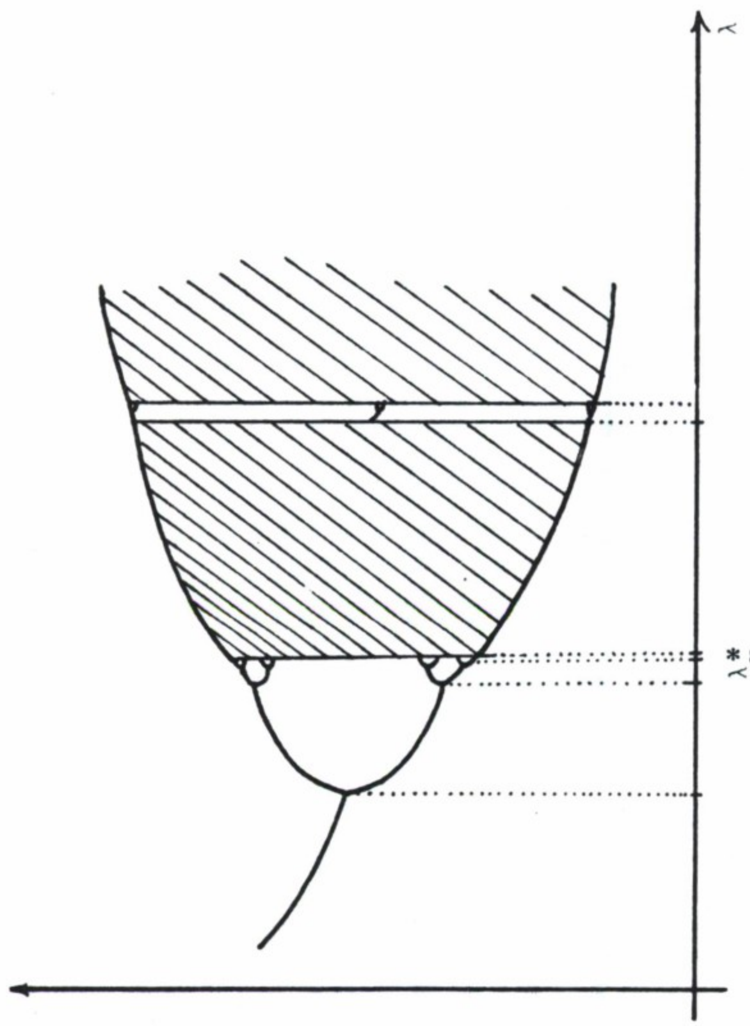

Figure 3

maps f_λ in which λ belongs to $[0,1]$. For each λ in that interval, f_λ maps the interval $[a_\lambda, b_\lambda]$ into itself, is C^1-unimodal with a unique critical point x^*_λ in (a_λ, b_λ) and $f_\lambda(x^*_\lambda) = b_\lambda$. We assume that a_λ and b_λ depend continuously on λ, as well as f_λ and its derivatives. More precisely, for any sequence λ_n that tends to λ in $[0,1]$, then $a_n = a_{\lambda_n}$ and $b_n = b_{\lambda_n}$ tend to a_λ and b_λ respectively, while for any sequence $x_n \in [a_n, b_n]$ that converges to $x \in [a_\lambda, b_\lambda]$, the sequences $f_{\lambda_n}(x_n)$ and $f'_{\lambda_n}(x_n)$ converge to $f_\lambda(x)$ and $f'_\lambda(x)$, respectively.

We shall say that the family if full if

1. for $\lambda = 0$, one has $f_o(b_o) > x^*_o$. In that case, as one can easily verify, all iterates $f^j_o(x^*_o) = f^{j-1}_o(b_o)$ belong to the interval $[f_o(b_o), b_o]$ for $j \geqq 1$.

2. for $\lambda = 1$, one has $f^2_1(x^*_1) < x^*_1$ and $f^3_1(x^*_1) < x^*_1$.

Then we have

<u>Theorem 7</u>: Consider a full one-parameter family of C^1-unimodal maps indexed by λ in $[0,1]$. Then

1) Given an arbitrary $k \geqq 2$, the set of parameters λ for which the map f_λ has a superstable cycle of period k is closed and nonempty. Given such a λ, there is an open interval around λ such that $f_{\lambda'}$ has a stable cycle of period k for all λ' in this interval.

2) Let λ^*_j be the first value of the parameter λ for which a superstable cycle of period 2^j obtains for $j \geqq 1$. Then the sequence λ^*_j increases with j and converges to some value $\lambda^*_\infty < 1$ as j tends

to $+\infty$. For each λ in $[0,\lambda^*_\infty)$, all cycles of the map f_λ have a period that is a power of 2 or are fixed points. The critical point x^*_λ of f_λ is attracted to one of these.

3) If superstable cycles of periods 2^j and $2^{j'}$ with $j' > j + 1$ occur respectively for the values λ and λ' in $[0,\lambda^*_\infty)$, then a superstable cycle of period 2^i with $j' > i > j$ must appear for some value in the open interval determined by λ and λ'.

Proof: As a preliminary remark, CE require that $a_\lambda = -1$, $b_\lambda = 1$, $x^*_\lambda = 0$ for all λ, but the proofs of the results we shall use, employ only simple continuity arguments that do not depend upon these specific assumptions. Second, our assumptions imply that the itinerary of b_o, denoted $K(f_o)$, is R^∞, while that of b_1, denoted $K(f_1)$ starts RLL ... (itineraries are defined in CE, p. 64)

1) According to (CE, Theorem III.1.1), every maximal admissible sequence A satisfying $K(f_o) < A < K(f_1)$ occurs as the itinerary $K(f_\lambda)$ of b_λ for some λ in $(0,1)$ (admissible sequences are defined in CE, p. 64, the ordering between admissible sequences is defined in CE, p. 65-66, while maximal sequences are defined in CE, p. 71). In fact, it follows from the proof of this theorem (see CE, p. 175) that the set of such λ's is nonempty and closed provided that $A \neq (BR)^\infty$ and $A \neq (BL)^\infty$.

Choose now an integer $k \geqq 2$, and consider a maximal sequence BC in which the sequence B contains $k - 1$ elements, such that

$$R^\infty < BC < RLLL \ldots$$

Given k, the set of such sequences is necessarily finite. It is not difficult to verify that is nonempty. As a matter of fact, we have

Lemma 8: One has[12/]

$$RLL\ldots > \ldots > RLR^{i-3}\,C > (RLR^{i-2})^{\infty} > RLR^{i-1}\,C > \ldots$$

$$> \ldots > R{*}RLR^{i-3}\,C > R{*}(RLR^{i-2})^{\infty} > R{*}RLR^{i-1}\,C > \ldots$$

$$> \ldots > R^{*n}{*}RLR^{i-3}\,C > R^{*n}{*}(RLR^{i-2})^{\infty} > R^{*n}{*}RLR^{i-1}\,C > \ldots$$

$$> \ldots > R^{*(m+1)}{*}RC > R^{*(m+1)}{*}R^{\infty} > R^{*m}{*}RC > \ldots > RC > R^{\infty}$$

in which $i \geqq 3$ is odd, $n \geqq 1$ and $m \geqq 1$ are arbitrary.

Proof: If one ignores the finite sequences in this series of inequalities, what has been written is simply the translation of (CE, Theorems II.2.8 and II.2.9). What we have done is just to insert these finite sequences. Now the first line of inequalities and the fact that the first sequence appearing on the second line satisfies

$$R{*}RLC = RLRRRC < RLR^{i-3}\,C$$

for every odd integer $i \geqq 3$ is readily verified by inspection. Then all the lines of inequalities except the last one follow by induction from the fact that R* is monotone among the set of maximal itineraries (see CE, Theorem II.2.5). The last line is in fact (CE, Lemma II.2.12) combined with their Theorems II.2.8 and II.2.9. Q.E.D.

Thus given the integer $k \geqq 2$, the set of maximal sequences BC in

which the sequence B contains $(k - 1)$ elements, and such that

$$R^{\infty} < BC < RLL \ldots$$

is nonempty and finite (it is nonempty since one may take $BC = R^{*n} * RLR^{i-3}\ C$ if $k = 2^n \cdot i$ with $n \geqq 0$ and $i \geqq 3$, i odd, and $BC = R^{*m} * RC$ if $k = 2^{m+1}$ with $m \geqq 0$). Therefore the set of values of λ such that the itinerary $K(f_\lambda)$ of b_λ coincides with one such BC is closed and nonempty. To show the first part of 1), it suffices to remark that for any λ, the itinerary $K(f_\lambda)$ of b_λ is maximal (see CE, p. 71) and that f_λ has a superstable cycle of period k if and only if $K(f_\lambda)$ coincides with one of the BC mentioned above. The last of 1) is a straightforward continuity argument that is left to the reader.

2) Lemma II.2.2 in CE states that the sequences appearing in the last line of inequalities in Lemma 8 above are consecutive among the maximal sequences ("consecutive" is defined in the statement of Lemma II.2.2 in CE). It follows then from (CE, theorem III.1.1) that the itinerary $K(f_{\lambda_j^*})$ of $b_{\lambda_j^*}$ is $R^{*(j-1)} * RC$, and that $\lambda_i^* > \lambda_j^* > 0$ whenever $i > j$ (otherwise λ_j^* would not be the minimum value of λ for which a superstable cycle of period 2^j obtains). The sequence λ_j^* converges thus towards $\lambda_\infty^* \leqq 1$. By another application of Lemma II.2.2 and Theorem III.1.1 in CE, one gets that for any λ in $[0,\lambda_\infty^*)$, the itinerary $K(f_\lambda)$ of b_λ is one of the sequences appearing in the last line of inequalities in Lemma 8 above. Since there are values of λ in $[0,1]$ such that f_λ has a superstable cycle with a period that

differs from a power of 2, one must have $\lambda_\infty^* < 1$. Next remark that the sequences appearing in the last line of inequalities in Lemma 8 are periodic with a period that is a power of two (see CE, Remark 1, p. 79). Thus for any λ in $[0,\lambda_\infty^*)$, the critical point x_λ^* is attracted to a periodic orbit the period of which is a power of 2 (see Lemmas II.3.1 and II.3.2 in CE). If the map f_λ has another cycle, then the itinerary $I(x)$ of the rightmost point x of the periodic orbit is maximal (see CE, p. 71) and satisfies $I(x) \leqq K(f_\lambda)$ (see CE, Lemma II.1.3). Again from CE, Lemma II.2.2, this itinerary $I(x)$ is one of the sequences appearing in the last line of inequalities of Lemma 8 that are less than or equal to $K(f_\lambda)$, or it is the sequence L^∞ (see Lemma II.2.1 in CE). This periodic orbit has a period that is a power of 2 or is a fixed point of f_λ.

3) This statement follows again from the fact that the sequences appearing in the last line of inequalities in Lemma 8 are consecutive among the maximal sequences, and from (CE, Theorem III.1.1). Q.E.D.

<u>Theorem 9</u>: Consider a full one-parameter family of C^1-unimodal maps indexed by λ in $[0,1]$, and assume that for each λ, the map f_λ (or one of its topological conjugates g_λ) satisfies S1, S2, S3, S4" and S5. Then

1) for any λ in $[0,\lambda_\infty^*)$, the map f_λ has a (unique) weakly stable periodic orbit

2) there is an uncountable set of values of λ in $(\lambda_\infty^*,1]$ for which f_λ has no weakly stable periodic orbit.

Proof:

1) We have seen when proving 2) of Theorem 7, that for any λ in $[0,\lambda^*_\infty)$, the itinerary $K(f_\lambda)$ of b_λ was one of the sequences that appeared in the last line of inequalities of Lemma 8. Since any one of these sequences is periodic, the result follows from Proposition 3.

2) By the argument of CE, pp. 184-85, there is an uncountable set of values of λ for which the extended itinerary of b_λ is not periodic. By Proposition 3, for each such λ, f_λ has no weakly stable cycle. From 1), all these values of λ must belong to $(\lambda^*_\infty,1]$. Q.E.D.

Remarks:

1. Under the assumptions of Theorem 9, it can be shown that there is an uncountable set of values of λ in $(\lambda^*_\infty,1]$ for f_λ has sensitivity to initial conditions, see (CE, Proposition III.2.1).

2. A good deal of recent research aimed at showing that the set of values of λ for which f_λ has no weakly stable cycle (has sensitivity to initial conditions) (has an absolutely continuous invariant probability measure) has positive Lebesque measure. For more information, see CE, Section I.5 and III.2.

3. For practically all families for which bifurcation diagrams have been computed, one observes striking numerical regularities. For instance, if λ_j is the value for which there is a bifurcation from a cycle of period 2^j to a period 2^{j+1}, then $(\lambda_j - \lambda_{j-1})/(\lambda_{j+1} - \lambda_j)$ tends very rapidly, as j diverges to $+\infty$, to some number $\delta = 4.66920 \ldots$, that seems independent of the family f_λ under consideration. For a discussion of this and related points, and a

theorem that gives a partial mathematical explanation of this "empirical" phenomenon, see CE, Sections I.6 and III.3. For an extension to families of maps on $\mathbb{R}^m$, with $m \geqq 2$, see CE, Section III.4.

Footnotes

1/ For applications to economics, see the references cited above. For an excellent review of the applications in other fields, see May [1976].

2/ Another, more recent review which presents essentially the same facts but from a slightly different point of view is provided by J. Guckenheimer and P. Holmes [1983].

3/ CE requires that $a = -1$, $x^* = 0$, $b = 1$. However, none of their arguments depend upon that specification and they are valid for the case at hand. We shall use that fact repeatedly without any further explicit reference.

4/ Singer's result is actually more general, since he showed that the number of stable cycles of an arbitrary map with a negative Schwarzian derivative is bounded above by the number of its critical points.

5/ CE use "stable" to denote what we call "weakly stable."

6/ These facts have been confirmed to me privately by Pierre Collet.

7/ To be precise, Proposition II.5.7 in CE is correct under assumptions S1, S2, S3, S4, S5 provided that $f(b)$ is not a fixed point of f satisfying $f'(f(b)) = 1$ (this fact has also been confirmed to me privately by Pierre Collet). This circumstance is however ruled out by S4". We may therefore apply their Proposition II.5.7 when $f(b) \leqq x^*$.

8/ This property is generic, i.e., it holds on a Baire set (a countable intersection of open and dense sets) in the space of once differentiable maps with the C^1-topology, if one discards the assumption that f has a negative Schwarzian derivative.

9/ See, e.g. Parthasarathy ([1967], Theorem 9.1). That book contains also the definition of the weak convergence of probability measures.

10/ The general definition of topological conjugacy requires only that h is an increasing homeomorphism (h is onto, continuous, increasing and h^{-1} is continuous also). The discussion that follows is in fact valid in this general case, we stick nevertheless to the differentiable case to simplify the presentation.

11/ Diagrams of this type are numerous in the literature. See CE, p. 26, May [1976]. Such bifurcation diagrams have been obtained in economic models by Dana and Malgrange [1981], Grandmont [1983], Jensen and Urban [1982].

12/ The product $A*B$ is defined in CE, p. 72, the notation A^{*n} is introduced in CE, p. 76.

References

Benhabib, J. and R.H. Day [1981], "Rational Choice and Erratic Behaviour," Review of Economic Studies, 48, pp. 459-472.

Benhabib, J. and R.H. Day [1982], "A Characterisation of Erratic Dynamics in the Overlapping Generations Model," Journal of Economic Dynamics and Control, 4, pp. 37-55.

Collet, P. and J.-P. Eckmann [1980], Iterated Maps on the Interval as Dynamical Systems, Birkhauser, Boston.

Dana, R.A. and P. Malgrange [1981], "The Dynamics of a Discrete Version of a Growth Cycle Model," CEPREMAP Working Paper, forthcoming in Analysing the Structure of Econometric Models, J.P. Ancot (Ed.), M. Nijhoff, Amsterdam.

Day, R.H. [1982], "Irregular Growth Cycles," American Economic Review, 72, pp. 406-414.

Day, R.H. [1983], "The Emergence of Chaos from Classical Economic Growth," Quarterly Journal of Economics, 98, pp. 201-13.

Grandmont, J.M. [1983], "On Endogenous Competitive Business Cycles," CEPREMAP DP No. 8316. Also available as a IMSSS Technical Report, Economics, Stanford University, and a EHEC Technical Report, Economics, University of Lausanne.

Guckenheimer, J. and P. Holmes [1983], Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Math. Sciences Series No. 42, Springer.

Jensen, R.U. and R. Urban [1982], "Chaotic Price Behaviour in a Nonlinear Cobweb Model," Yale University, mimeo.

Li, T. and J.A. Yorke [1975], "Period Three Implies Chaos," American Mathematical Monthly, 82, pp. 985-992.

May, R.B. [1976], "Simple Mathematical Models with Very Complicated Dynamics," Nature, 261, pp. 459-467.

Parthasarathy, K.R. [1967], Probability Measures on Metric Spaces, Academic Press, New York.

Rand, D. [1978], "Exotic Phenomena in Games and Duopoly Models," Journal of Mathematical Economics, 5, pp. 173-184.

Sarkovskii, A.N. [1964], "Coexistence of Cycles of a Continuous Map of the Line Into Itself," Urk. Mat. Z., 16, pp. 61-71.

Singer, D. [1978], "Stable Orbits and Bifurcations of Maps of the Interval," SIAM Journal of Applied Mathematics, 35, p. 260.

Stefan, P. [1977], "A Theorem of Sarkovskii on the Existence of Periodic Orbits of Continuous Endomorphisms of the Real Line," Comm. Math. Phys., 54, pp. 237-248.

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